How to Easily Calibrate the Dial of a Pendulum Scale

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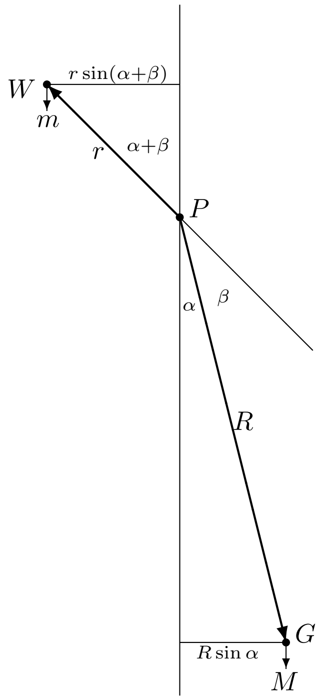
The mathematical development presented here is neither complicated (it uses just regular high-school maths) nor new (I am sure that others have thought of the same before), but it is useful for anyone who needs to calibrate the dial of a pendulum scale.

We consider a gravity based scale in which a pendulum with the center of gravity G rotates freely around a fixed pivot P. The weight to be measured acts vertically on a well defined point W on the pendulum, e.g. by hanging down or using a weight tray mounted on an axis that is kept vertical with a parallel lever.



Let's first define the following variables:

m	*(to be measured)*	Mass of the weight to be measured
r	*(known)*	Distance from pivot P to point W where the weight m is attached
M	*(unknown constant)*	Mass of the pendulum including any fixed part of the weight holding mechanism (hook, tray, ...) that is acting vertically at W, but excluding the mass of the weight to be measured m.
R	*(unknown constant)*	Distance from pivot point P to the center of gravity of the pendulum G
α	*(unknown)*	Deflection angle in function of the attached weight m
β	*(measurable)*	Angle of weight point W from the vertical at pivot P when $m = 0$

In the following, we'll show that we can calibrate the deflection function $\alpha(m)$ without knowing the details of the scale geometry nor its own weight – simply by measuring two angles:

1. The angle β can be measured when no weight is on the scale, as in this case the yet unknown point G must necessarily be vertically below P. So β is the angle between $W - P$ and the vertical.
2. For an arbitrary know weight m_1, measure the resulting deflection angle α_1. Best choose a weight which which results in a deflection angle that corresponds roughly to the middle of the scale dial.

Now lets look at the mathematics...

The scale is at equilibrium if the momentum of the weight equals the momentum of the scale pendulum:

$$mr\sin(\alpha+\beta) = MR\sin\alpha \tag{1}$$

The unknown product MR can be obtained from the measured deflection angle α_1 and corresponding weight m_1 by solving (1) for MR:

$$MR = m_1 r\frac{\sin(\alpha_1+\beta)}{\sin\alpha_1} \tag{2}$$

Dividing the above equation by r, we get

$$\frac{MR}{r} = m_1\frac{\sin(\alpha_1+\beta)}{\sin\alpha_1} \tag{3}$$

Note that MR/r is a constructional constant of the particular pendulum. So the left hand side of the above equation should always result in the same value, regardless of the test weight used. *(This also opens the possibility to reduce the measuring error by using several measurements (m_i, α_i) and then calculate $\frac{MR}{r}$ as the average of the obtained values.)*

Using the fact that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ we can rewrite (1) as

$$mr(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = MR\sin\alpha \tag{4}$$

which can be solved for α with

$$(MR - mr\cos\beta)\sin\alpha = mr\sin\beta\cos\alpha \tag{5}$$

leading to

$$\tan\alpha = \frac{mr\sin\beta}{MR - mr\cos\beta} = \frac{m\sin\beta}{\frac{MR}{r} - m\cos\beta} \tag{6}$$

We now can substitute MR by the measurement (m_1,α_1) using (2)

$$\tan\alpha = \frac{m\sin\beta}{m_1\frac{\sin(\alpha_1+\beta)}{\sin\alpha_1} - m\cos\beta} \tag{7}$$

Note that r has now dropped completely out of the formula. By expanding the fraction by $\sin\alpha_1$, we get

$$\tan\alpha = \frac{m\sin\beta\sin\alpha_1}{m_1\sin(\alpha_1+\beta) - m\cos\beta\sin\alpha_1} \tag{8}$$

Now we use again the fact that $\sin(\alpha_1+\beta) = \sin\alpha_1\cos\beta + \cos\alpha_1\sin\beta$ as well as $\tan\alpha_1 = \sin\alpha_1/\cos\alpha_1$ to obtain

$$\tan\alpha = \frac{m\tan\beta\sin\alpha_1}{m_1(\sin\alpha_1 + \cos\alpha_1\tan\beta) - m\sin\alpha_1} \tag{9}$$

Therefore, the calibrated function for the deflection angle is

$$\alpha(m) = \arctan\left(\frac{m\tan\beta\sin\alpha_1}{m_1(\sin\alpha_1 + \cos\alpha_1\tan\beta) - m\sin\alpha_1}\right) \tag{10}$$

In conclusion we have shown that for computing $\alpha(m)$ we only need the two measured angles α_1 and β plus the weight of the used mass m_1 - thus avoiding the need to know the constructional parameters r, R and M!